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THE FORCING DETOUR COTOTAL DOMINATION NUMBER OF A GRAPH

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Abstract

Let S be a detour cototal dominating set of G. A subset $D \subseteq S$ is called a forcing subset of S if S is the unique minimum detour cototal dominating set containing D. The minimum cardinality D is the forcing detour cototal domination number of S and is denoted by $f_{\gamma dct}(S)$, is the cardinality of a minimum forcing subset of S. The forcing detour cototal domination number of G, denoted by $f_{\gamma dct}(S)$, is $f_{\gamma dct}(G) = \min\{f_{\gamma dct}(S)\}$, where the minimum is taken over all γ_{dct} -sets S in G. Some general properties satisfied by this concept are studied. It is shown that for every pair a, b of integers with $0 \leq a \leq b, b \geq 2$, there exists a connected graph G such that $f_{\gamma dct}(G) = a$ and $\gamma_{dct}(G) = b$. Where $\gamma_{dct}(G)$ is the detour cototal dominating number of G.

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1. Introduction

For a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by m and n respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies we refer to [2, 7]. For vertices u and v in a graph G, the detour distance $D(u, v)$ is the length of a detour distance $D(u, v)$ is the length of a longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ detour. It is known that the detour distance is a metric on the vertex set $V(G)$. A subgraph obtained from graph G by vertex deletion only is an induced subgraph of G. If X is the set of deleted vertices, the induced subgraph is denoted by $G - X$ with $Y = V(G)/X$, the induced subgraph is denoted as $G[Y]$ and called the subgraph of G induced by vertex set Y. A vertex x is said to lie on a $u - v$ detour P if x is a vertex of $u - v$ detour path P including the vertices u ad v. A set $S \subseteq V(G)$ is called a detour set of G if every vertex v in G lies on a detour joining a pair of vertices of S. The closed detour interval $I_D[u, v]$ consists of u, v and all vertices in some $u - v$ detour of G. For $S \subseteq V(G)$, $I_D[S] = \cup_{u, v \in S} I_D[u, v] = V(G)$. A subset S of V of a graph G is called a detour set if $I_D[S] = V(G)$. detour number $d_n(G)$ of G is the minimum cardinality taken over all detour sets in G. These concepts were studied by Chartrand [5, 6.10]. A set $S \subseteq V(G)$ is called a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The domination number, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ-set of G. The domination concept was studied in [8]. A dominating set S of G is a cototal dominating set if every vertex $v \in V \setminus S$ is not an isolated vertex in the induced subgraph $\langle V \setminus S \rangle$. The cototal domination number $\gamma_{ct}(G)$ of G is the minimum cardinality of a cototal dominating set. The cototal domination number of a graph was studied in [11, 12,]. A set $S \subseteq V$ is said to be a detour cototal dominating set of G, if S is both detour set and cototal dominating set of G. The detour cototal domination number of G is the minimum cardinality among all detour cototal dominating sets in G and denoted by $\gamma_{dct}(G)$. A detour cototal dominating set of minimum cardinality is called the γ_{dct}-set of G. The detour cototal domination number of a graph was studied in [9,]. The following theorems are used in the sequel.

Theorem 1.1 [10]. *Every end vertex of G belongs to every detour dominating set of G.*

Theorem 1.2 [10]. *For the non-trivial tree,* $\gamma_{dct}(G) = k$, *where k is the number of end vertices of G.*

2. The Forcing Detour Cototal Domination Number of a Graph

Even though every connected graph contains a minimum detour cototal dominating sets, some connected graph may contain several minimum detour cototal dominating sets. For each minimum detour cototal dominating set S in a connected graph there is always some subset T of S that uniquely determines S as the minimum detour cototal dominating set containing T such "forcing subsets" are considered in this section. The forcing concept was studied in [1, 3, 9].

Definition 2.1. Let S be a detour cototal dominating set of G. A subset $D \subseteq S$ is called a forcing subset of S if S is the unique minimum detour cototal dominating set containing D. The minimum cardinality D is the forcing detour cototal domination number of S and is denoted by $f_{\gamma dct}(S)$, is the cardinality of a minimum forcing subset of S. The forcing detour cototal domination number of G, denoted by $f_{\gamma dct}(S)$, is $f_{\gamma dct}(G) = \min\{f_{\gamma dct}(S)\}$, where the minimum is taken over all γ_{dct}-sets S in G.

Example 2.2. For the graph G of Figure 2.1, $S_1 = \{v_1, v_4, v_7, v_8\}$ and $S_2 = \{v_2, v_4, v_7, v_8\}, S_3 = \{v_1, v_6, v_8, v_{10}\}, S_4 = \{v_1, v_5, v_8, v_9\}, S_5 = \{v_2, v_6, v_8, v_{10}\}$, $S_6 = \{v_3, v_5, v_8, v_9\}$, $S_7 = \{v_1, v_5, v_8, v_{10}\}$, $S_8 = \{v_1, v_6, v_8, v_9\}$, $S_9 = \{v_2, v_5, v_8, v_{10}\}$, $S_{10} = \{v_2, v_6, v_8, v_9\}$ are the only ten γ_{dct}-sets of G, such that $f_{\gamma dct}(S_1) = f_{\gamma dct}(S_2) = 2$ and $f_{\gamma dct}(S_i) = 3$ for $3 \leq i \leq 10$. So that $f_{\gamma dct}(G) = 2$ and $\gamma_{dct}(G) = 4$.

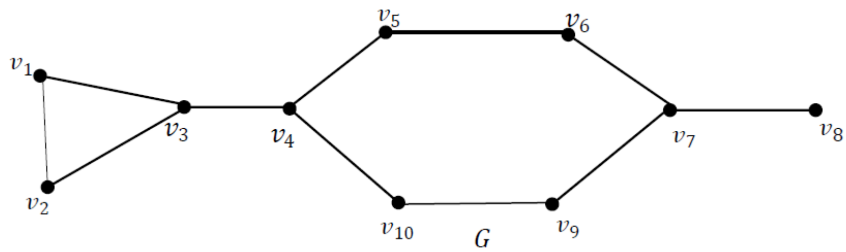

Figure 2.1.

The following result follows immediately from the definitions of the detour cototal domination number and the forcing detour cototal domination number of a connected graph G.

Theorem 2.3. *For every connected graph* G, $0 \leq f_{\gamma dct}(G) \leq \gamma_{dct}(G)$.

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the Star graph $G = K_{1, n-1}$, $(n \geq 3)$, $S = V(G)$ is the unique γ_{dct}-set of G so that $f_{\gamma dct}(G) = 0$. Also for the Cycle $G = C_4$ with $V(G) = \{v_1, v_2, v_3, v_4\}$, $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_2, v_3\}$, $S_3 = \{v_3, v_4\}$, $S_4 = \{v_4, v_1\}$ are the only four γ_{dct}-sets of G, such that $f_{\gamma dct}(G) = \gamma_{dct}(G) = 2$. Also the bounds in Theorem 2.3 can be strict. For the graph G given in Figure 2.1, $\gamma_{dct}(G) = 4$ and $f_{\gamma dct}(G) = 2$. Thus $0 < f_{\gamma dct}(G) < \gamma_{dct}(G)$.

Theorem 2.5. *Let G be a connected graph. Then*

(a) $f_{\gamma dct}(G) = 0$ *if and only if G has a unique minimum* γ_{dct}*-set.*

(b) $f_{\gamma dct}(G) = 1$ *if and only if G has at least two minimum* γ_{dct}*-sets, one of which is a unique minimum* γ_{dct}*-set containing one of its elements and*

(c) $f_{\gamma dct}(G) = \gamma_{dct}(G)$ *if and only if no* γ_{dct}*-set of G is the unique minimum* γ_{dct}*-set containing any of its proper subsets.*

Definition 2.6. A vertex v of a connected graph G is said to be a detour cototal dominating vertex of G if v belongs to every γ_{dct}-set of G.

Example 2.7. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_4, v_7\}$,

$S_2 = \{v_2, v_4, v_7\}$ and $S_3 = \{v_3, v_4, v_7\}$ are the only three γ_{dct}-sets of G, such that $\{v_4, v_7\}$ is the set of all detour cototal dominating vertices of G.

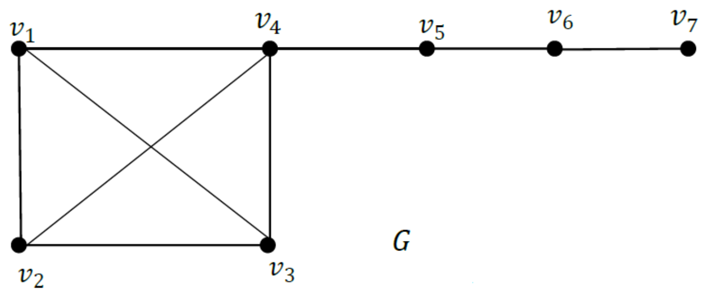

Figure 2.2.

Remark 2.8. Every end vertex of G is a detour cototal dominating vertex of G. Infact there are detour cototal dominating vertices which are not end vertices of G, For the graph given in Figure 2.2, v_4 is a detour cototal dominating vertex of G, which is not an end vertex of G.

Theorem 2.9. *Let G be a connected graph and W be the set of all detour cototal dominating vertices of G. Then* $f_{\gamma dct}(G) \leq \gamma_{dct}(G) - |W|$.

Remark 2.10. The bounds in Theorem 2.9 is sharp. For the graph G given in Figure 2.2, $|W| = 2, \gamma_{dct}(G) = 3$ and $f_{\gamma dct}(G) = 1$. Thus $f_{\gamma dct}(G) = \gamma_{dct}(G) - |W|$. Also the bounds in Theorem 2.9 is strict, for the graph G given in Figure 2.1, $|W| = 3$. $\gamma_{dct}(G) = 4$ and $f_{\gamma dct}(G) = 2$. Thus $f_{\gamma dct}(G) = \gamma_{dct}(G) - |W|$.

Theorem 2.11. *For the complete bipartite graph* $G = K_{r,s} (1 \leq r \leq s)$,

$$f_{\gamma dct}(G) = \begin{cases} 0, & if\ r = 1,\ s \geq 2 \\ 2, & if\ 2 \leq r \leq s. \end{cases}$$

Proof. For $r = 1$ and $s \geq 2$, $S = V(G)$ is the unique γ_{dct}-set of G so that $f_{\gamma dct}(G) = 0$. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the bipartite sets of G. Let $u \in U$ and $w \in W$. Then $S = \{u, w\}$ is a unique γ_{dct}-

set of G. Since $r \geq 2, f_{\gamma dct}(G) \geq 2.$ Since this is true for all $u \in U$ and $w \in W, S$ is not a unique γ_{dct}-set of G containing u or w. Therefore, $f_{\gamma dct}(S) = 2.$ Since this is true for all γ_{dct}-sets of $G, f_{\gamma dct}(G) = 2.$ ■

Theorem 2.12. *For the wheel graph* $G = K_1 + C_{n-1}(n \geq 5), f_{\gamma dct}(G) = 1.$

Proof. Let x be the central vertex of G and C_{n-1} be $v_1, v_2, \ldots, v_{n-1}, v_1.$ Then $S_i = \{x, v_i\}\ (1 \leq i \leq n-1)$ is a γ_{dct}-set of G such that $f_{\gamma dct}(S_i) = 1(1 \leq i \leq n-1)$ so that $f_{\gamma dct}(G) = 1.$ ■

Theorem 2.13. *For the Fan graph,* $G = K_1 + P_{n-1}(n \geq 5), f_{\gamma dct}(G) = 1.$

Proof. Let $V(K_1) = \{x\}$ and $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}.$ Then $S_1 = \{x, v_1\}$ and $S_2 = \{x, v_{n-1}\}$ are the only two a γ_{dct}-sets of G such that $f_{\gamma dct}(S_1) = f_{\gamma dct}(S_2) = 1.$ So that $f_{\gamma dct}(G) = 1.$ ■

Theorem 2.14. *For the helm graph* $G = H_r, f_{\gamma dct}(G) = 0,$ *for* $n \geq 6.$

Proof. Let S be the set of end vertices and the cut vertices of G. Then S is the unique γ_{dct}-set of G so that $f_{\gamma dct}(G) = 0.$ ■

Theorem 2.15. *For the graph* $G = K_{1, a+1} + e, f_{\gamma dct}(G) = 1.$

In view of Theorem 2.3, we have the following realization result.

Theorem 2.16. *For every pair* a, b *of integers with* $0 \leq a < b, b \geq 2,$ *there exists a connected graph G such that* $f_{\gamma dct}(G) = a$ *and* $\gamma_{dct}(G) = b.$

Proof. For $a = 0, b \geq 2,$ let $G = K_{1, a-1}.$ Then by Theorem 1.2 and 2.11, $\gamma_{dct}(G) = b$ and $f_{\gamma dct}(G) = a.$ So, let $2 \leq a \leq b.$

Case (i). $2 \leq a = b.$

Let $P_i : u_i, v_i(1 \leq i \leq a)$ be a path with three vertices. Let G be a graph obtained from $P_i(1 \leq i \leq a)$ by introducing a vertex x and joining x with each $u_i, v_i(1 \leq i \leq a).$ The graph G is given in Figure 2.3.

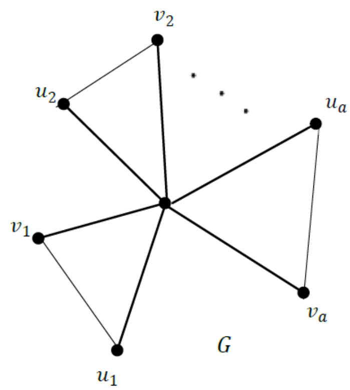

Figure 2.3

First we prove that, $\gamma_{dct}(G) = a$. It is easily observed that every γ_{dct}-set of G contains at least one vertex from each component of $G - x$ and so $\gamma_{dct}(G) \geq a$. Let $S = \{v_1, v_2, \ldots, v_a\}$. Then S is a γ_{dct}-set of G so that $\gamma_{dct}(G) = a$. Next we prove that $f_{\gamma dct}(G) = a$. By Theorem 2.3, $f_{\gamma dct}(G) \leq a$. Let $H_i = \{u_i, v_i\}(1 \leq i \leq a)$. Then every γ_{dct}-set of G contains at least one vertex from each $H_i(1 \leq i \leq a)$. Therefore every γ_{dct}-set of S is of the form $S = \{C_1, C_2, \ldots, C_a\}$ where $C_i \in H_i(1 \leq i \leq a)$. Since this is true for all γ_{dct}-set of G, $f_{\gamma dct}(G) = a$.

Case (ii). $2 \leq a < b$

Let $P : x, y$ be a path on two vertices and $P_i : u_i, v_i(1 \leq i \leq a)$ be a copy of path on two vertices. Let H be a graph obtained from P and $P_i(1 \leq i \leq a)$ by joining x with each u_i and $v_i(1 \leq i \leq a)$. Let G be the graph obtained from H by introducing new vertices $z_1, z_2, \ldots, z_{b-a}$ and joining y with each $v_i(1 \leq i \leq b - a)$. The graph G is shown in Figure 2.4.

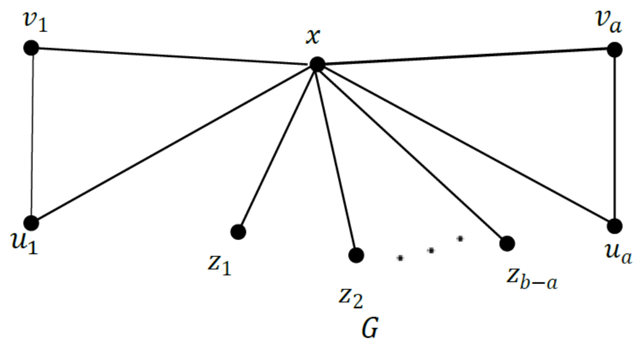

Figure 2.4

First we prove that, $\gamma_{dct}(G) = b$. Let $Z = \{z_1, z_2, \ldots, z_{b-a}\}$ be the set of end vertices of G. By Theorem 1.1, Z is a subset of every detour cototal dominating set of G. Let $H_i = \{u_i, v_i\} (1 \leq i \leq a)$. Then it is easily observed that every detour cototal dominating set contains at least one vertex from each $H_i (1 \leq i \leq a)$ and So that, $\gamma_{dct}(G) \geq b - a + b = b$. Let $S = Z \cup \{u_1, u_2, \ldots, u_a\}$.

Next we prove that $f_{\gamma dct}(G) = a$. Since every detour cototal dominating set contains z, It follows from Theorem 2.9, $f_{\gamma dct}(G) \leq \gamma_{dct}(G) - |Z|$ $= b - (b - a) = a$. Now since $\gamma_{dct}(G) = b$ and every γ_{dct}-set of G contains Z, it is easily seen that every γ_{dct}-set of G is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\}$ where $C_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then there exist an edge $e_j (1 \leq j \leq a)$ such that $e_j \notin T$. Let f_j be an edge of H_i distinct from e_j. Then $W = (S - \{e_j\}) \cup \{f_j\}$ is a detour cototal dominating set of G properly containing T. Thus W is not the unique γ_{dct}-set containing T. Thus T is not the forcing subset of S. This is true for all minimum detour cototal dominating sets of G and so it follows that $f_{\gamma dct}(G) = a$.

Conclusion

In this paper we studied the concept of forcing detour cototal domination number of graph. In future studies, this same concept is applied for the other graph operations.

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